RAPIDLY CONVERGENT SERIES FOR THE WEIERSTRASS ZETA-FUNCTION AND THE KRONECKER FUNCTION

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ABSTRACT. We present expressions for the Weierstrass zeta-function and related elliptic functions by rapidly convergent series. These series arise as triple products in the A_∞-category of an elliptic curve.

1. FORMULAS

In this section we derive our formulas by classical means. In the next section we'll explain how one can guess these formulas from the computation of certain triple products on elliptic curves.

1.1. **Weierstrass zeta-function.** Let L be a lattice in $\mathbb{C}$, ω_1, ω_2 be generators of L such that $\mathrm{Im}(\overline{\omega_1}\omega_2) > 0$. The Weierstrass zeta-function is defined by the series

$$\zeta(x, L) = \frac{1}{x} + \sum_{\omega \in L-0} \left(\frac{1}{x+\omega} - \frac{1}{\omega} + \frac{x}{\omega^2}\right)$$

for $x \in \mathbb{C} \setminus L$. One has

$$\zeta(x+\omega, L) - \zeta(x, L) = \eta(\omega)$$

for any $\omega \in L$, where $\eta(\omega)$ is a constant. We denote $\eta_i = \eta(\omega_i)$ for $i = 1, 2$. It is well-known that

$$\eta_i = 2\zeta\left(\frac{\omega_i}{2}\right) \tag{1.1}$$

for $i = 1, 2$. Following Hecke (see [1]) for any $x = x_1\omega_1 + x_2\omega_2 \in \mathbb{C} - L$ we set

$$Z(x, L) = \zeta(x, L) - x_1\eta_1 - x_2\eta_2$$

(here x_1 and x_2 are real).

Theorem 1. *For any $x \in \mathbb{C} - L$ one has the following identity*

$$Z(x, L) = \sum_{\omega \in L} \frac{\exp(-\frac{\pi}{a(L)}|\omega + x|^2)}{\omega + x} - \sum_{\omega \in L-0} \frac{\exp(-\frac{\pi}{a(L)}|\omega|^2 + 2\pi i E_L(\omega, x))}{\omega} \tag{1.2}$$

where $a(L) = \mathrm{Im}(\overline{\omega_1}\omega_2)$ is the area of $\mathbb{C}/L$,

$$E_L(x, y) = \frac{\mathrm{Im}(\overline{x}y)}{a(L)} = \frac{\overline{x}y - x\overline{y}}{2ia(L)}$$

is the symplectic form on $\mathbb{C}$ (considered as a real space) associated with the oriented lattice L.

Let $\mathcal{S}(\mathbb{C})$ be the Schwarz space of $\mathbb{C}$. For any $\varphi \in \mathcal{S}(\mathbb{C})$ we define its *symplectic* Fourier transform by the formula

$$\hat{\varphi}(y) = \int_{x \in \mathbb{C}} \varphi(x) \exp(2\pi i E_L(y, x)) d_L x$$

where $d_L x$ is the Haar measure on $\mathbb{C}$ normalized by the condition $\int_{\mathbb{C}/L} d_L x = 1$.

We will need the following simple lemma.

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Lemma 1.1. *For any $\varphi \in \mathcal{S}(\mathbb{C})$, any $x, y \in \mathbb{C}$ one has*

$$\sum_{\omega \in L} \varphi(\omega + x) \exp(-2\pi i E_L(\omega + x, y)) = \sum_{\omega \in L} \hat{\varphi}(\omega + y) \exp(-2\pi i E_L(\omega, x))$$

Proof. By Poincare summation formula the distribution $\delta_L = \sum_{\omega \in L} \delta_\omega$ is Fourier self-dual. Since the translation by x goes under Fourier transform to the multiplication by $\exp(-2\pi i E_L(?, x))$ the result follows. □

Proof of theorem 1. Let us denote

$$f(x) = \sum_{\omega \in L} \frac{\exp(-\frac{\pi}{a(L)}|\omega + x|^2)}{\omega + x} - \sum_{\omega \in L - 0} \frac{\exp(-\frac{\pi}{a(L)}|\omega|^2 + 2\pi i E_L(\omega, x))}{\omega}$$

where $x \in \mathbb{C} - L$. We claim that

$$\overline{\partial} f(x) = -\frac{\pi}{a(L)}.$$

Indeed, we have

$$\overline{\partial} f(x) = -\frac{\pi}{a(L)} \cdot \left(\sum_{\omega \in L} \exp(-\frac{\pi}{a(L)}|\omega + x|^2) - \sum_{\omega \in L - 0} \exp(-\frac{\pi}{a(L)}|\omega|^2 + 2\pi i E_L(\omega, x)) \right),$$

so our claim follows from Lemma 1.1 since the function $\exp(-\frac{\pi}{a(L)}|x|^2)$ goes to itself under the symplectic Fourier transform.

It follows that the function $g(x) = f(x) + \frac{\pi}{a(L)}\overline{x}$ is holomorphic on $\mathbb{C} - L$. Furthermore, looking at the series for f we immediately see that g has simple poles at all the lattice points $\omega \in L$ with residues equal to 1.

On the other hand, from the fact that the symplectic form E_L takes integer values on L one immediately derives that $f(x + \omega) = f(x)$ for any $\omega \in L$. Thus, we have

$$g(x + \omega) = g(x) + \frac{\pi}{a(L)}\overline{\omega}$$

for $\omega \in L$. The Legendre period relation

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$$

implies that there exists a constant $c \in \mathbb{C}$ such that

$$\eta_i = c\omega_i + \frac{\pi}{a(L)}\overline{\omega_i} \tag{1.3}$$

for $i = 1, 2$. It follows that $h(x) = g(x) - \zeta(x, L) + cx$ is a holomorphic function on $C - L$, periodic with respect to L. Comparing the polar parts of g and ζ at the lattice points we conclude that h is holomorphic on $\mathbb{C}$. Therefore, h is constant and we have

$$f(x) - \zeta(x, L) = -\frac{\pi}{a(L)}\overline{x} - cx + h.$$

From the definition of the constant c we derive that

$$x_1\eta_1 + x_2\eta_2 = cx + \frac{\pi}{a(L)}\overline{x}$$

where $x = x_1\omega_1 + x_2\omega_2$. Thus, we have

$$f(x) - \zeta(x, L) = -x_1\eta_1 - x_2\eta_2 + h.$$

It is easy to see that if $x \in \frac{1}{2}L - L$ then $f(x)$. Thus, substituting $x = \frac{\omega_1}{2}$ and using the identity (1.1) we derive that $h = 0$. □

Remarks. 1. It is not obvious that the right hand side of (1.2) is holomorphic in ω_1, ω_2. This fact is equivalent to the identity

$$\sum_{\omega\in L}(\omega+x)\exp(-\frac{\pi}{a(L)}|\omega+x|^2)=\sum_{\omega\in L}\omega\exp(-\frac{\pi}{a(L)}|\omega|^2+2\pi iE_L(\omega,x))$$

which can be easily deduced from Lemma 1.1.
2. It is well-known that the following Epstein's zeta function

$$\varphi_1(s,L,x)=\sum_{\omega\in L}\frac{1}{(\omega+x)|\omega+x|^{2s-1}}$$

defined for $\mathrm{Re}(s)>1$ extends to an entire function of s (for fixed $x\in\mathbb{C}-L$). It was shown by N. Katz (Cor. 3.2.24 of [2]) that

$$\varphi_1(\frac{1}{2},L,x)=Z(x,L)$$

for $x\in\mathbb{Q}L-L$. It is not clear whether there exists an expression for $\varphi_1(s,L,x)$ for arbitrary s similar to the one in (1.2).

Differentiating the identity (1.2) we obtain the following series for the Weierstrass $\wp$-function $\wp(x)=-\zeta'(x)$:

$$\wp(x)=-c+\sum_{\omega\in L}\frac{(1+\frac{\pi}{a(L)}|\omega+x|^2)\exp(-\frac{\pi}{a(L)}|\omega+x|^2)}{(\omega+x)^2}+\frac{\pi}{a(L)}\cdot\sum_{\omega\in L-0}\frac{|\omega|^2\exp(-\frac{\pi}{a(L)}|\omega|^2+2\pi iE_L(\omega,x))}{\omega^2} \tag{1.4}$$

where the constant c is determined from (1.3). Differentiating one more time we get

$$\wp'(x)=-\sum_{\omega\in L}\frac{(1+(1+\frac{\pi}{a(L)}|\omega+x|^2)^2)\exp(-\frac{\pi}{a(L)}|\omega+x|^2)}{(\omega+x)^3}+\frac{\pi^2}{a(L)^2}\cdot\sum_{\omega\in L-0}\frac{|\omega|^4\exp(-\frac{\pi}{a(L)}|\omega|^2+2\pi iE_L(\omega,x))}{\omega^3}. \tag{1.5}$$

1.2. **Kronecker function.** Let us consider the following holomorphic function in 3 variables τ, x, y, where $\mathrm{Im}(\tau)>0$, $0<\mathrm{Im}(x),\mathrm{Im}(y)<\mathrm{Im}(\tau)$:

$$F(x,y;\tau)=-\sum_{(m+\frac{1}{2})(n+\frac{1}{2})>0}\mathrm{sign}(m+\frac{1}{2})\exp(2\pi imn\tau+2\pi imx+2\pi iny)$$

where m,n are integers (our choice of sign is compatible with the notation in Zagier's paper [8], but our variables x and y differ from those used in [8] by the factor $2\pi i$). We call it the Kronecker function since Kronecker discovered (see [3]) the following remarkable identity:

$$F(x,y;\tau)=\frac{\theta'_{11}(0,\tau)}{2\pi i}\cdot\frac{\theta_{11}(x+y,\tau)}{\theta_{11}(x,\tau)\theta_{11}(y,\tau)} \tag{1.6}$$

where

$$\theta_{11}(x,\tau)=\sum_{n\in\mathbb{Z}}(-1)^n\exp(\pi i(n+\frac{1}{2})^2\tau+2\pi i(n+\frac{1}{2})x),$$

θ'_{11} is the derivative of $\theta_{11}(x,\tau)$ with respect to x. In particular, this identity gives a meromorphic continuation of F to $\mathfrak{H}\times\mathbb{C}^2$ with poles along the divisors $x\in L_\tau$, $y\in L_\tau$, where $L_\tau=\mathbb{Z}+\mathbb{Z}\tau$.

Theorem 2. *One has the following identity*

$$\begin{aligned}2\pi iF(x,y,\tau)=\exp(-\tfrac{\pi}{\mathrm{Im}\,\tau}x(y-\overline{y}))\cdot\textstyle\sum_{\omega\in L_\tau}\frac{\exp(-\frac{\pi}{\mathrm{Im}\,\tau}|\omega+x|^2-2\pi iE(\omega,y))}{\omega+x}+\\ \exp(-\tfrac{\pi}{\mathrm{Im}\,\tau}y(x-\overline{x}))\cdot\textstyle\sum_{\omega\in L_\tau}\frac{\exp(-\frac{\pi}{\mathrm{Im}\,\tau}|\omega+y|^2-2\pi iE(\omega,x))}{\omega+y}\end{aligned} \tag{1.7}$$

where $E=E_{L_\tau}$.

Proof. For a fixed τ let $f(x,y)$ be the function in the right hand side of (1.7). First we have to check that $f(x,y)$ is meromorphic in x and y. Let us denote $a = \mathrm{Im}(\tau)$. We have

$$\frac{\partial}{\partial \overline{x}} f = -\frac{\pi}{a} \cdot \exp(-\frac{\pi}{a} x(y-\overline{y})) \cdot \sum_{\omega \in L_\tau} \exp(-\frac{\pi}{a}|\omega + x|^2 - 2\pi i E_L(\omega, y)) +$$
$$\frac{\pi}{a} \cdot \exp(-\frac{\pi}{a} y(x-\overline{x})) \cdot \sum_{\omega \in L_\tau} \exp(-\frac{\pi}{a}|\omega + y|^2 - 2\pi i E_L(\omega, x)).$$

Thus, we have to prove that

$$\sum_{\omega \in L_\tau} \exp(-\frac{\pi}{a}|\omega + x|^2 - 2\pi i E_L(\omega + x, y)) = \sum_{\omega \in L_\tau} \exp(-\frac{\pi}{a}|\omega + y|^2 - 2\pi i E_L(\omega, x)).$$

But this follows easily from Lemma 1.1 since the function $\exp(-\frac{\pi}{a}|x|^2)$ is Fourier self-dual.

Next we observe that for any $\omega \in L$ one has

$$f(x+\omega, y) = \exp(-\frac{\pi}{a}(\omega - \overline{\omega})y) f(x,y).$$

Since $f(x,y) = f(y,x)$ we conclude that f has the same quasi-periodicity equations as F. Hence, f/F is periodic with respect to L_τ in both variables. Since both f and F have poles of the first order at $x \in L_\tau$ and $y \in L_\tau$ the only possible poles of f/F can come from zeroes of F. But the only zeroes of F are the zeroes of the first order along the divisor $x + y \in L_\tau$. On the other hand, one immediately checks that $f(x, -x) = 0$. Therefore, f/F is holomorphic, so it should be constant. Now the identity follows by the comparison of the residues of F and f at $x = 0$. $\square$

From identities (1.2) and (1.7) one immediately deduces the following result.

Corollary 1.2. *One has*

$$\left(2\pi i F(x,y,\tau) - \frac{1}{y}\right)|_{y=0} = \zeta(x, L_\tau) - x\eta_1$$

where as generators of L_τ we take $\omega_1 = 1$, $\omega_2 = \tau$.

Another way to express the relation between the function F and Weierstrass zeta-function is the following:

$$\pi i \cdot (F(x,y,\tau) + F(x,-y,\tau))|_{y=0} = \zeta(x, L_\tau) - x\eta_1.$$

This can be deduced either from the above corollary or using (1.6) and the formula

$$\frac{\theta_{11}'(x)}{\theta_{11}(x)} = \zeta(x, L_\tau) - x\eta_1$$

which can be seen from the decomposition of θ_{11} into an infinite product.

2. Explanation

Both the functions $Z(x,L)$ and $F(x,y,\tau)$ have nice modular properties. The modular forms $Z(x_1\omega_1 + x_2\omega_2, L)$ for fixed $x_1, x_2 \in \mathbb{Q}$ were considered by Hecke in [1] (he called them "Teilwerte" of the Weierstrass zeta-function). The modular equation for $F(x,y,\tau)$ can be found in [8]. The formulas (1.2) and (1.7) provide an alternative explanation of modularity but each of the series in the right hand side is non-holomorphic (only the difference of two such series is). In this section we show that this is related to the computation of certain triple products on elliptic curve using non-holomorphic data (namely, hermitian metrics). However, since the result doesn't depend on a choice of non-holomorphic data the resulting expressions are holomorphic in the modular parameter.

We refer to [4] for the general discussion of higher products on elliptic curve. The triple products related to the two series considered in the previous section are of the following type. Let L, M_1, M_2 be hermitian line bundles of degree 1 on a complex elliptic curve. Then one can consider a triple product

$$m_3 : H^0(E, M_1) \otimes H^1(E, L^{-1}) \otimes H^0(E, M_2) \to H^0(E, M_1M_2L^{-1}).$$

Recall that it is given by the formula

$$m_3(s_1, e, s_2) = \mathrm{pr}(Q(s_1e)s_2 - s_1Q(es_2)) \tag{2.1}$$

where $s_i \in H^0(E, M_i)$, $i = 1, 2$ the class $e \in H^1(E, L^{-1})$ is represented by a harmonic $(0, 1)$-form, $Q = \overline{\partial}^* G_{\overline{\partial}}$ where $G_{\overline{\partial}}$ is the Green operator corresponding to the laplacian $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ (where $\overline{\partial}^*$ is conjugate to $\overline{\partial}$ with respect to the hermitian metric), $\mathrm{pr} = \mathrm{id} - G_{\overline{\partial}}\Delta_{\overline{\partial}}$ is the harmonic projector. The formula (2.1) shows that in fact our triple product depends only on the operator Q acting on forms with values in M_1L^{-1} and M_2L^{-1}. Both these line bundles are of degree zero so (up to switching M_1 and M_2) the following three possibilities can occur:

(a) $M_iL^{-1} \not\simeq \mathcal{O}_E$ for $i = 1, 2$. Then we have $Q = (\overline{\partial})^{-1}$ in the formula (2.1) so this triple product doesn't depend on metrics. We will show that in this case m_3 is expressed in terms of the series appearing in the formula (1.7).

(b) $M_1L^{-1} \simeq \mathcal{O}_E$, $M_2L^{-1} \not\simeq \mathcal{O}_E$. In this case the operator Q depends on a hermitian metric on $\mathcal{O}_E$. However, there is a natural choice of a constant metric on $\mathcal{O}_E$ and it is easy to see that Q doesn't change if we rescale a metric by a constant. In this case we'll express m_3 in terms of the series from the formula (1.2).

(c) $M_1L^{-1} \simeq M_2L^{-1} \simeq \mathcal{O}_E$. One can easily see that in this case $m_3 = 0$.

To compute triple products in the cases (a) and (b) we represent our elliptic curve in the form $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. We choose L to be the line bundle on E such that the theta-function

$$\theta(z) = \theta(z, \tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i\tau n^2 + 2\pi inz)$$

descends to a section of L. Thus, the pull-back of L to $\mathbb{C}$ is canonically trivialized. For $u \in \mathbb{C}$ let us denote by $L(u)$ the line bundle t_u^*L where $t_u : E \to E$ is the translation by u (note that a choice of $u \in \mathbb{C}$ induces a trivialization of the pull-back of $L(u)$ to $\mathbb{C}$). We define the hermitian metric on $L(u)$ by the formula

$$\langle f, g\rangle_{L(u)} = \int_{C/\mathbb{Z}+\mathbb{Z}\tau} f(x)\overline{g(x)} \exp(-2\pi a(x_2^2 + 2x_2u_2)dx_1dx_2$$

where we use real coordinates x_1, x_2 defined by $x = x_1 + x_2\tau$, so that $u = u_1 + u_2\tau$ and we denote $a = \mathrm{Im}\,\tau$. With respect to this metric one has

$$||t_u^*\theta||^2 = \frac{1}{\sqrt{2a}} \exp(2\pi au_2^2).$$

As line bundles M_1 and M_2 we take $L(u)$, $L(v)$ for some $u, v \in \mathbb{C}$, so that we have natural choice of sections $s_1 = t_u^*\theta$, $s_2 = t_v^*\theta$. As a harmonic $(0, 1)$-form representing a non-trivial class $e \in H^1(E, L^{-1})$ we take

$$\alpha = \frac{\pi\sqrt{2}}{\sqrt{a}}\overline{\theta(x)} \exp(-2\pi ax_2^2)d\overline{x}.$$

Our computation will be based on the following formula which was proven in [4] (it is equivalent to eq. (2.2.1) of [4], on the other hand, it can be deduced from Prop. 4.1 of [6]):

$$\begin{aligned}&\theta(x + y)\overline{\theta(x + z)} \exp(-2\pi a(x_2^2 + 2x_2z_2)) =\\ &\tfrac{1}{\sqrt{2a}} \textstyle\sum_{m,n}(-1)^{mn} \exp(-\frac{\pi}{2a}(|m\tau - n|^2 + 2(m\overline{\tau} - n)y - 2(m\tau - n)\overline{z} + (y - \overline{z})^2))\varphi_{y-z,m,n}(x)\end{aligned} \tag{2.2}$$

where we denote

$$\varphi_{w,m,n}(x) = \exp(2\pi i(mx_1 + (n - w)x_2)),$$

the summation is over $(m,n)\in\mathbb{Z}^2$. Note that $(\varphi_{w,m,n})_{(m,n)\in\mathbb{Z}^2}$ descend to the orthonormal basis of sections on $L(w)L^{-1}$. We have

$$\overline{\partial}\varphi_{w,m,n}=\frac{\pi}{a}(m\tau-n+w)\varphi_{w,m,n}d\overline{x}.$$

In the case $w\notin\mathbb{Z}+\mathbb{Z}\tau$ this allows us to compute $Q=(\overline{\partial})^{-1}$ in terms of coefficients with the basis $\varphi_{w,m,n}d\overline{x}$. In the case $w=0$ the operator Q still coincides with $(\overline{\partial})^{-1}$ on $\varphi_{0,m,n}d\overline{x}$ for $(m,n)\neq(0,0)$. On the other hand, $\varphi_{0,0,0}=1$ and we have

$$\overline{\partial}^*(d\overline{x})=0,$$

hence, $Q(d\overline{x})=0$.

Let us first consider the case (a). Then we have

$$m_3(t_u^*\theta,\alpha,t_v^*\theta)=\mathrm{pr}(h_ut_v^*\theta-h_vt_u^*\theta)$$

where we denote $h_w=Q(t_w^*\theta\cdot\alpha)$. Using formula (2.2) we get that for any $w\notin\mathbb{Z}+\mathbb{Z}\tau$ one has

$$h_w=\sum_{m,n}a_{m,n}(w)\varphi_{w,m,n}$$

where

$$a_{m,n}(w)=\frac{(-1)^{mn}\exp(-\frac{\pi}{2a}(|m\tau-n|^2+2(m\overline{\tau}-n)w+w^2))}{m\tau-n+w}. \tag{2.3}$$

By definition the above triple product is proportional to $t_{u+v}^*\theta$ so the computation reduces to calculating the coefficient

$$\frac{\langle m_3(t_u^*\theta,\alpha,t_v^*\theta),t_{u+v}^*\theta\rangle}{||t_{u+v}^*\theta||^2}=\sqrt{2a}\exp(-2\pi a(u_2+v_2)^2)\cdot\left(\langle h_ut_v^*\theta,t_{u+v}^*\theta\rangle-\langle h_vt_u^*\theta,t_{u+v}^*\theta\rangle\right). \tag{2.4}$$

Now by definition we have

$$\langle h_ut_v^*\theta,t_{u+v}^*\theta\rangle=\int_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}h_u(x)\theta(x+v)\overline{\theta(x+u+v)}\exp(-2\pi a(x_2^2+2x_2(u_2+v_2)))dx_1dx_2=$$
$$\langle h_u,t_{u+v}^*\theta\cdot\overline{t_v^*\theta}\exp(-2\pi a(x_2^2+2x_2v_2))\rangle$$

where the last scalar product is taken with respect to the metric on $L(u)L^{-1}$. Applying (2.2) we get

$$t_{u+v}^*\theta\cdot\overline{t_v^*\theta}\exp(-2\pi a(x_2^2+2x_2v_2))=\sum_{m,n}b_{m,n}(u,v)\varphi_{u,m,n}$$

where

$$b_{m,n}(u,v)=\frac{1}{\sqrt{2a}}(-1)^{mn}\exp(-\frac{\pi}{2a}(|m\tau-n|^2+2(m\overline{\tau}-n)(u+v)-2(m\tau-n)\overline{v}+(u+v-\overline{v})^2)). \tag{2.5}$$

Since $\varphi_{u,m,n}$ is an orthonormal system we derive

$$\langle h_ut_v^*\theta,t_{u+v}^*\theta\rangle=\sum_{m,n}a_{m,n}(u)\overline{b_{m,n}(u,v)}.$$

Substituting the expressions (2.3) and (2.5) and simplifying we obtain

$$\sqrt{2a}\exp(-2\pi a(u_2+v_2)^2)\cdot\langle h_ut_v^*\theta,t_{u+v}^*\theta\rangle=\exp(\frac{\pi}{a}u(v-\overline{v}))\cdot\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau}\frac{\exp(-\frac{\pi}{a}|\omega+u|^2+2\pi iE(\omega,v))}{\omega+u}$$

where E is the symplectic form associated with the oriented lattice $\mathbb{Z}+\mathbb{Z}\tau$. Substituting this into the equation (2.4) we obtain that the coefficient of $m_3(t_u^*\theta,\alpha,t_v^*\theta)$ with $t_{u+v}^*\theta$ is equal to

$$\exp(\frac{\pi}{a}u(v-\overline{v}))\cdot\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau}\frac{\exp(-\frac{\pi}{a}|\omega+u|^2+2\pi iE(\omega,v))}{\omega+u}-\exp(\frac{\pi}{a}v(u-\overline{u}))\cdot\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau}\frac{\exp(-\frac{\pi}{a}|\omega+v|^2+2\pi iE(\omega,u))}{\omega+v}.$$

According to the identity (1.7) this expression is equal to $2\pi iF(u,-v,\tau)$. In [4] it was shown that the series defining $F(u,-v,\tau)$ appears as the corresponding triple product in the Fukaya category of the torus, so we can view the identity (1.7) as a manifestation of the homological mirror symmetry[1].

In the case (b) we set $u=0$ and slightly modify the above computation. Namely, we have

$$h_0=Q(\theta\cdot\alpha)=\sum_{(m,n)\neq(0,0)}a_{m,n}(0)\varphi_{0,m,n}$$

where $a_{m,n}(0)$ are still given by the formula (2.3). The formula (2.4) still holds for $u=0$, so we obtain that the coefficient of the triple product in this case is equal to

$$\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau,\omega\neq0}\frac{\exp(-\frac{\pi}{a}|\omega|^2+2\pi iE(\omega,v))}{\omega}-\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau}\frac{\exp(-\frac{\pi}{a}|\omega+v|^2)}{\omega+v}.$$

According to the identity (1.2) this is equal to $-Z(v,\mathbb{Z}+\mathbb{Z}\tau)$. Thus, considering triple products above one can discover the identity (1.2) as follows. One starts with the identity (1.7) which follows from the homological mirror symmetry for elliptic curve. Then one tries to pass to the limit as $u\to0$. From the expression of the Weierstrass zeta-function as the logarithmic derivative of sigma-function it is easy to guess that the limit of $F(u,-v)$ should be related to $\zeta(v,\mathbb{Z}+\mathbb{Z}\tau)$. On the other hand, it should be modular, so one naturally arrives to considering $Z(v,\mathbb{Z}+\mathbb{Z}\tau)$.

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[1] In [5] we worked with the RHS of the identity (1.7) without presenting an explicit series for it.